ECON 2201 Utility Maximization Examples

Robert Szarka

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1 A Well-Behaved Utility Function

Rob's preferences for books (B) and compact discs (C) are represented by the utility function $U = BC^2$. If books cost \$10 each, CDs costs \$5 each, and Rob's weekly income is \$90, how many of each will he consume this week?

First, let's graph the budget constraint (with B on the horizontal axis and C on the vertical axis). Equating total expenditure to total income gives us $10B + 5C = 90$, and solving this equation for C gives us $C = 18 - 2B$. Note that the slope of -2 gives us the relative price of books in terms of CDs: each book we purchase costs 2 CDs. The vertical intercept of 18 shows that Rob can purchase 18 CDs if he devotes his entire income to CDs, while the horizontal intercept of 10 shows that he can purchase 10 books if he devotes his entire income to books.

Second, let's take a look at a typical indifference curve derived from Rob's utility function. Holding utility constant at some level, say $\bar{U}$, and solving his utility function for C, we obtain

$C = \frac{\sqrt{\bar{U}}}{\sqrt{B}}$

(Of course, we ignore any negtative roots. Also, note that this equation is valid only when $B \neq 0$ and $C \neq 0$.) Since $\bar{U}$ is constant, C clearly decreases with B, so we have a downward-sloping indifference curve. If we choose, say, $\bar{U} = 400$ and plot some representative points, we also see that it has the convex shape that we expect from a "well-behaved" utility function[1].

Is $\bar{U} = 400$ the best that Rob can do given \$90 of income? Clearly not, since there are feasible bundles (within or on the budget constraint) that have more of at least one good and at least as much of the other compared to those on the indifference curve. To find the best bundle for Rob to consume,

[1] Recall our informal definition: an indifference curve is *convex* if the secant line joining two points on the curve lies above the curve.

then, we need to equate the *marginal rate of substitution* and the *marginal rate of transformation*; that is, we set

$$\frac{MU_B}{MU_C} = \frac{P_B}{P_C} \tag{1}$$

where[2]. Substituting the given marginal utilities and prices, we have $C = 4B$. We then substitute this relation into our budget constraint to get $10B + 5(4B) = 90$ and solve to find that $B = 3$. Finally, using either equation we can find that $C = 12$, so we have that the optimal bundle for Rob to consume, given his income and the prices of the two goods, is 3 books and 12 CDs.

Note that, for example, the bundle $(4, 10)$ costs \$90 and lies on the $\bar{U} = 400$ indifference curve. Let's check to make sure $(3, 12)$ is, in fact, better. The utility from this bundle is $U = (3)(12^2) = 432$, so, yes, we it is. We could use the same approach to compare our optimal bundle to other feasible bundles.

[2] As Michael noted in class, this utility function does not exhibit *diminishing marginal utility* for either good. That is, the marginal utility of, say, books does not decrease as Rob consumes more books (holding the number of CDs consumed constant). Although diminishing marginal utility is an intuitively-appealing feature for most goods, and Marshall himself assumed it in his work, it's neither a necessary nor sufficient condition for convex indifference surfaces or downward-sloping demand curves. In the framework of the theory we're developing here, the key requirement is that the marginal rate of substitution declines as Rob consumes more books (as indeed it does) $MU_B = C^2$ and $MU_C = 2BC$. This is equivalent to saying that the absolute value of the "slope" of the indifference curve declines as we move along the horizontal axis. For a discussion of this issue, see *e.g.* George J. Stigler. *The Theory of Price*. Third Edition. New York: The Macmillan Company, 1966, pp. 51-53, 341.

2 Perfect Substitutes

Omer's preferences for apples (A) and bananas (B) are represented by $U = A + 2B$. *If apples cost \$2, bananas cost \$1, and Omer's weekly income is \$30, how many of each will he consume this week?*

Here we have a case of *perfect substitutes*, where indifference curves (with A on the horizontal axis) aren't strictly convex but are instead straight lines given by $B = \frac{\bar{U}}{2} - \frac{\bar{A}}{2}$. This is problematic.

Consider the alternate, "bang-for-the-buck" form of equation (1) above:

$$\frac{MU_A}{P_A} = \frac{MU_B}{P_B} \tag{2}$$

which equates the marginal utility per dollar of the two goods. Since $MU_A = 1$ and $MU_B = 2$, we have

$\frac{MU_A}{P_A} = \frac{1}{2} < 2 = \frac{MU_B}{P_B}$

for any combination of apples and bananas. Therefore, Omer's best course is to specialize in the consumption of bananas; that is, he chooses the bundle $(0, 30)$, obtaining $U = 60$. Were he to consume, say, 2 fewer bananas and spend the money on an apple instead, his utility would be only $U = 57$, making him worse off. Another way of saying this is that, in general, for perfect substitutes no unique *interior solution* (consuming positive quantities of both goods) can be found, so we can't use the procedure in the first example above. Instead, we may obtain a *corner solution* (spending the entire budget on one good and consuming none of the other).

Worse, were the price of bananas exactly twice the price of apples, the slope of the budget line would be the same as the slope of the indifference curves, and the answer would be indeterminate. For example, were the price of apples \$1 and the price of bananas \$2, any combination of apples and bananas that cost \$30 would give Omer the same $U = 30$.

Finally, note that were the relative price of bananas to increase even further, Omer would switch from specializing in the consumption of bananas to specializing in the consumption of apples (*i.e.* at the other "corner"). For example, if apples cost \$2 and bananas cost \$6, Omer will choose the bundle $(15, 0)$.

3 Perfect Complements

Professor Segerson's preferences for coffee (C) and donuts (D) are represented by $U = \min(C, D)$. If coffee costs \$5, donuts cost \$5, and her weekly income is \$100, how many of each will she consume this week?

Here we have a case of *perfect complements*, with right-angled indifference curves[3]. Recall that utility functions with this relationship between the two goods violate our axiom that "more is better". Unfortunately, we can't use the approach in the first example above for a function like this, either[4].

Although the tangent line to the indifference curve isn't defined at the vertex, it will be true that utility will be maximized for this utility function where the budget line passes through the vertex of an indifference curve. Consider a bundle like $(11, 9)$, which costs \$100 (so it lies on the budget line) and yields $U = 9$. Were Professor Segerson to consume slightly less coffee and more donuts, she could increase her utility. She could continue increasing her utility by substituting donuts for coffee up to the bundle $(10, 10)$ with utility $U = 10$. At this point, consuming less coffee and more donuts would result in lower utility.

[3]In case you're unfamiliar with functions of this form, our utility function says that the level of utility Professor Segerson achieves is equal to whichever is lower, the quantity of coffee or the quantity of donuts she consumes. If she consumes 2 cups of coffee and 1 donut, $U = 1$. But if she consumes 2 cups of coffee and 2 donuts, $U = 2$.

[4]What is the MRT at the vertex of one of these indifference curves? You can't divide by zero, so it's undefined.