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Sent: Monday, January 18, 2010 3:59 PM
To: UDOC
Subject: Information Delivery Services Request 48396

Request Date: 1/18/2010 3:53:05 PM
Need before: 06/30/2010
ILLiad Transaction #: 48396
Patron: Robert Szarka

Title: Scientific American
Volume / issue: /
Date: 1966 January
Pages: 105-110
Author: M. Shinbrot
Article/Chapter Title: Fixed-Point Theorems
ISSN:

____ Volume or Issue not found on shelf

____ Title not found on shelf as it is cited above

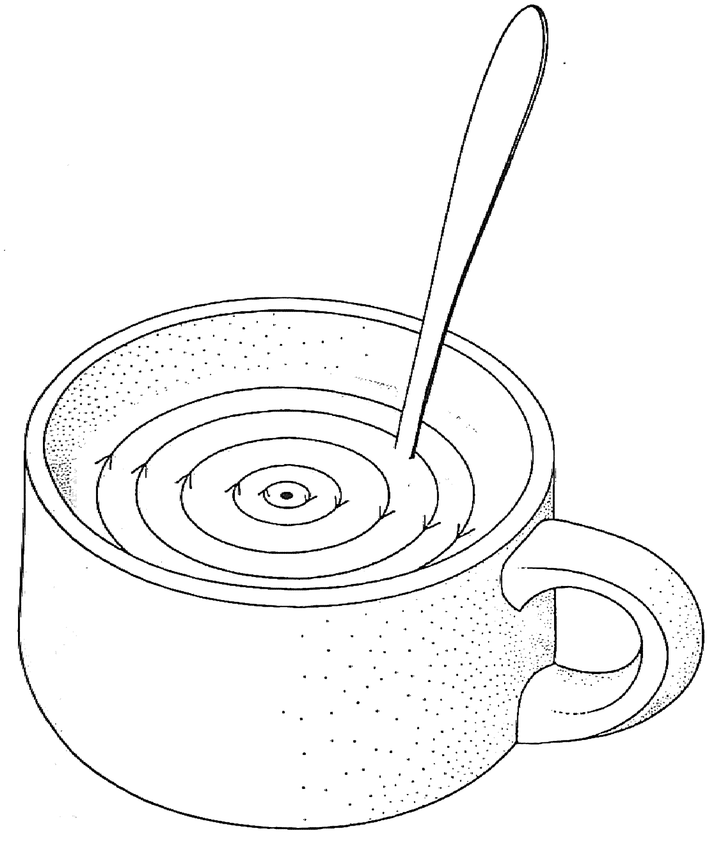
FIXED-POINT THEOREMS

These somewhat implausible laws state that points must reappear in their original positions when the surfaces on which they lie undergo certain deformations. Their practical uses are numerous

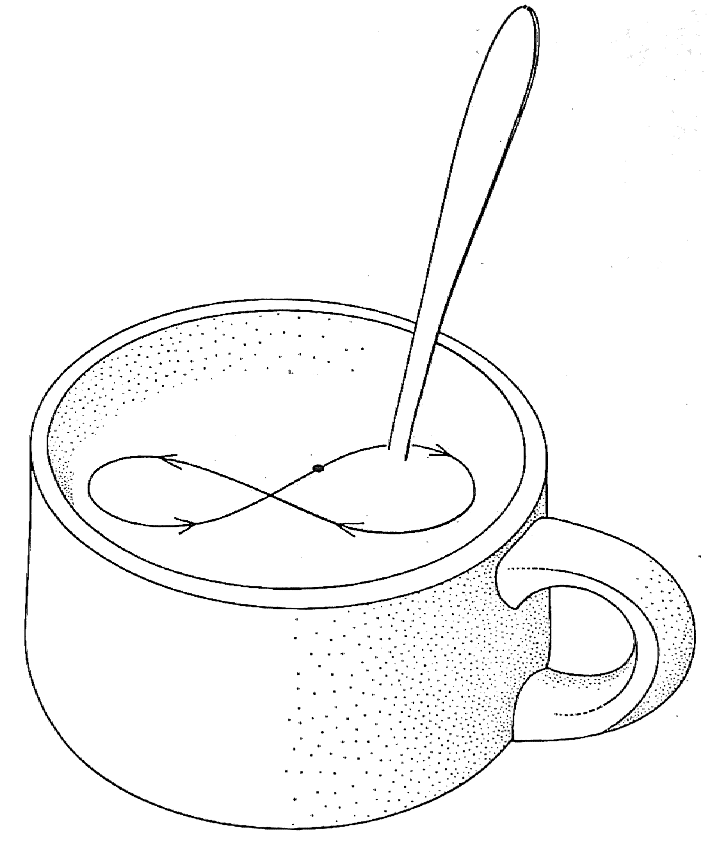
by Marvin Shinbrot

If you mark a series of points on a rubber band and then stretch it, the order in which the points appear does not change. This is an intuitively acceptable conclusion of topology: the study of properties that persist when geometric figures are bent, stretched, twisted or otherwise continuously deformed. Other topological facts are not so clear; their validity seems intuitively unacceptable. In this intriguing category are the fixed-point theorems, a group of results concerning points that reappear exactly in their original positions after the surfaces on which they lie have been deformed.

An example will serve to introduce them. Suppose we stir a cup of coffee, in any way and for any length of time but gently enough so that the surface is never disrupted. (As they say in cookbooks, "Stir, do not whip.") According to one of the simplest fixed-point theorems, when we have finished stirring and the motion of the liquid has stopped, at least one point on the surface will be back where it started! Such a point is called a fixed point. A particle at the exact center of the surface would be the fixed point in the simplest case: when the liquid is swirled only in circles. Usually the motion of stirred coffee is more complicated, with any particle susceptible to being moved to any position on the surface. The relevant fixed-point theorem, first proved by



CONTINUOUS DEFORMATION of a geometric surface is represented by the gentle swirling of coffee in which the thin film of cream on top is never disrupted. Here the coffee is being swirled in such a way that the particle in the exact center does not move.



FIXED-POINT THEOREM states that no matter how the surface of the coffee is continuously deformed, there will always be a point on the surface in the position it occupied at the start. This theorem does not stipulate which point is fixed at any instant in time.

the Dutch mathematician L. E. J. Brouwer, does not specify which point remains fixed but only that one or more points must do so.

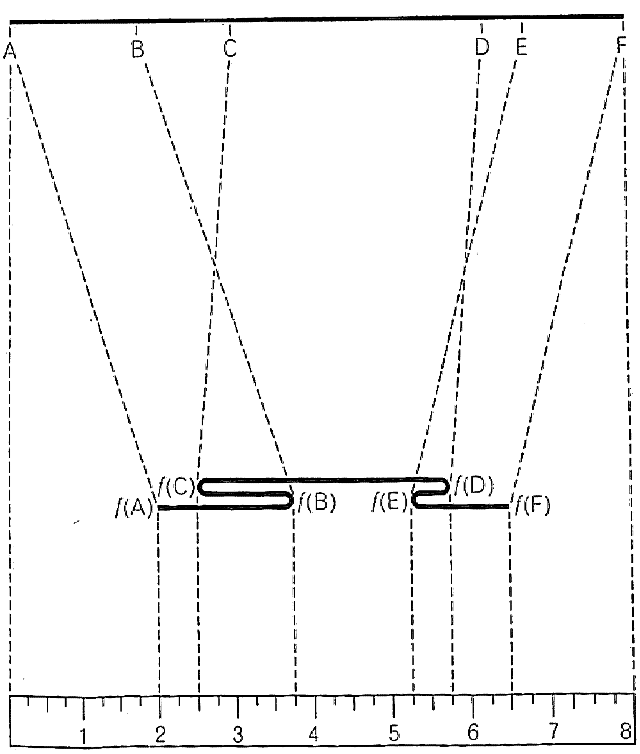
Consider another application of Brouwer's theorem. If this page of *Scientific American* were torn out, crumpled and folded in any way (but not torn) and then placed back on the magazine in such a way that no part of it extended beyond the edges of its original position, then at least one of the points on the crumpled page would lie directly above the spot it originally occupied. This fact, guaranteed by the Brouwer theorem, strikes many people as even more surprising than the certainty of a fixed point on the surface of the coffee. To the mathematician, however, it is more readily explained because the crumpling of a page is a simpler deformation than the swirling of coffee; the paper cannot be stretched, whereas the distance between two points on the surface of the coffee can easily change.

In order to understand how the proof of a fixed-point theorem might be constructed, it is simplest to look not at a two-dimensional surface such as the surface of the coffee or the sheet of paper but at a one-dimensional surface exemplified by a piece of string. Suppose we stretch a string to its full length so that it forms a straight line and then place it on a table. Next we fold the string any number of times and shift it around within the confines of the line made by the straight string. It can now be shown that a point on the string has returned to the exact spot it occupied before the manipulation and is therefore a fixed point. This is the one-dimensional version of the Brouwer theorem.

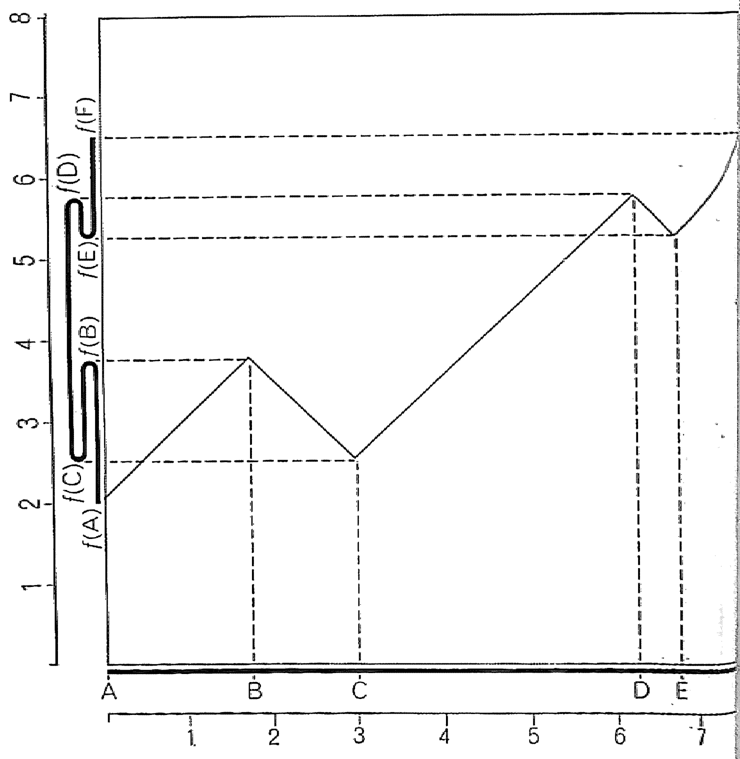
The theorem is proved by representing both the original string and the folded string as curves on a graph, comparing the curves and demonstrating that they intersect at some point [*see illustrations on these two pages*]. To begin, we measure the original straight string. We call the left end zero and specify each point on the string by its distance in inches from the left end. If the string is, say, eight inches long, we can speak of the point at the far right as "point eight." By the same token the position of each point on the folded string can be specified by its new distance from the left end of the string. If a point originally four inches from the left has been moved as a result of the deformation to three inches from the left, its new position is designated simply as "point three."

In this way we define a function, which can be denoted by $f(x)$. The value of this function at any point on the string is the number representing the position to which that point has been moved. Thus if point four is moved to point three, the value of f at four is three; in symbols $f(4) = 3$. To say that some point has not been moved—that is, to say that some point is a fixed point—is just a geometrically appealing way of saying that the equation $f(x) = x$ has a solution.

Now we construct a familiar Cartesian plane and graph the function $f(x)$. In this plane the horizontal axis designates the distance of each point from the left when the string was in its original position, and the vertical axis designates the distance of each point from the left after the string has been folded. On such a graph the point shifted from four to three can be plotted as a point with the coordinates four (on the horizontal axis) and three (on the vertical axis). When all the points on the folded string are plotted in this way, the curve connecting them is a mathematical representa-



FOLDING OF A STRING is a continuous deformation of a one-dimensional surface. Points on an eight-inch string (*top*) assume new positions when the string is folded (*middle*), with point A, for example, moving to point designated $f(A)$. A fixed-point theorem states that some point $f(P)$ on the folded string must be as far from the left of the ruler (*bottom*) as point P was on the straight string.



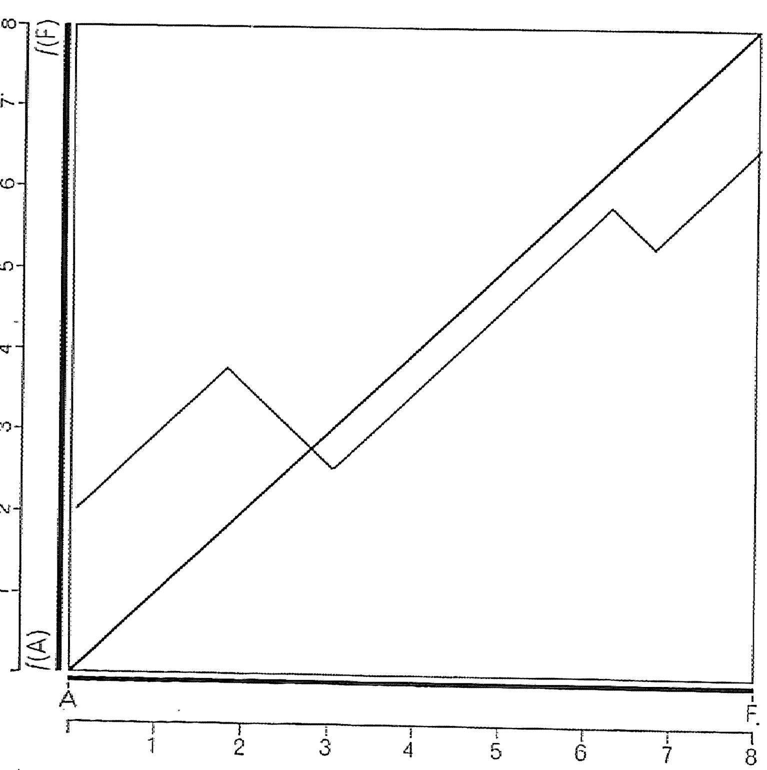
DESCRIPTION OF FOLDED STRING in illustration at left is provided by the jagged curve on this graph. Horizontal axis designates the distance of a point, in inches, from the left end of the original string. Vertical axis designates distance of a point on the folded string from same point on the original. Thus point C has the coordinates three (horizontal axis) and 2.5 (vertical axis).

tion of the physical folding of the string. This curve may be extremely complicated, but it has two nice and particularly significant properties. It lies entirely within one quadrant of the Cartesian plane; indeed, it lies within the square having zero to eight on the horizontal axis as its base. This is ensured by the fact that the deformed string was never moved off the original eight-inch segment; therefore the function $f(x)$ describing the deformation can be neither negative nor greater than eight. Moreover, the curve is continuous; since the string was not broken in the process of deformation, there are no breaks in the curve describing that deformation. These two properties of the curve suffice, as we shall show, to guarantee that the string has a fixed point.

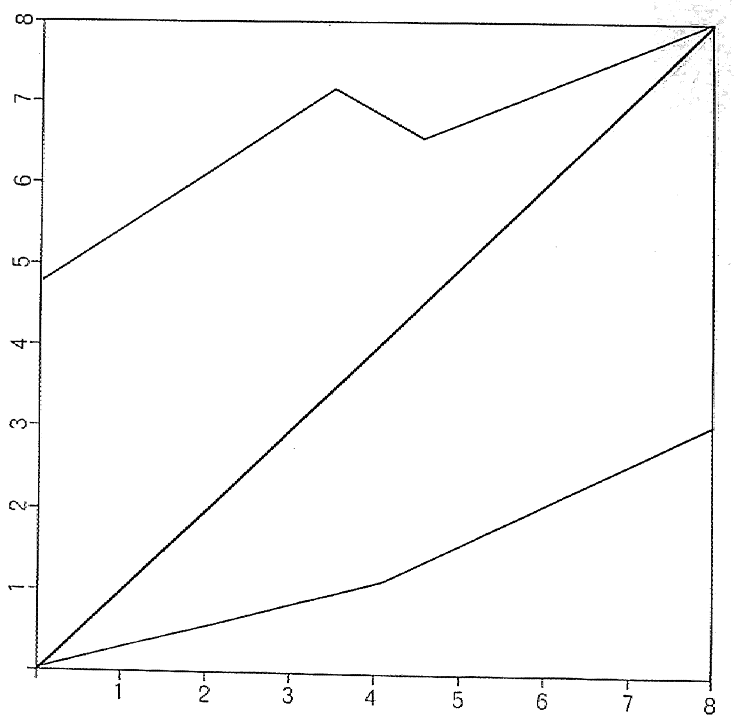
We know how to represent the folded string as a curve on a graph; we now have to demonstrate that if the original, undeformed string were also represented as a curve, the two curves *must* intersect. This is not difficult to show. Assume that we have picked up the straight string and returned it, still straight, to its original position. Even though it has not changed shape, we can consider that it has undergone a deformation and we can plot the function corresponding to this deformation. If we plot the "new" distances from the left end against the old, we get points that are equidistant from the two axes—points with coordinates one and one, two and two and so forth. When we connect these points, we have in fact drawn the diagonal of the square built on the base of zero to eight on the horizontal axis.

Now, recall the curve that represents the folded string. It must by definition begin at zero (the left side of our square) and end at eight (the right side). It also must lie between zero and eight on the vertical axis and can have no breaks. To get from one side of the square to the other it is necessary that this curve cross, or at least touch the diagonal. The only way for the curve representing the folded string not to cross the diagonal is for it to begin at the lower left corner of the square or end at the upper right corner. The first case, however, merely implies that point zero is a fixed point, and the second that point eight is fixed. Therefore in all cases there is some point of intersection between the two curves and thus a fixed point on the deformed string. This would hold true, incidentally, even if, instead of the string, we had used an elastic material such as rubber, provided only that the deformed piece was not broken and was replaced so as not to lie outside the position occupied by the undeformed piece. The only difference that the use of rubber would make is that the curve representing the stretched piece need not consist of line segments—as the representation of the folded string must—but may have a curved shape.

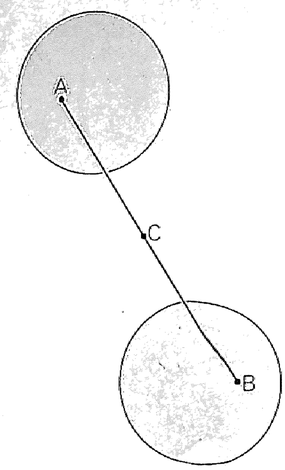
The form of the Brouwer theorem that applies to two-dimensional surfaces would also hold if, instead of the surface of coffee in a cup, we were considering an infinitely elastic circular piece of rubber. We can transform such a rubber disk by stretching and folding it in various ways, making sure only that the disk is not torn and that it is replaced within the original circumference. The proof of the two-dimensional version of the Brouwer theorem is most elegant. We first consider a disk and assume that, contrary to the theorem, no point on it remains fixed after a deformation; it is then possible to show that this assumption is untenable. The steps of the proof (which holds not only for a disk but also for a

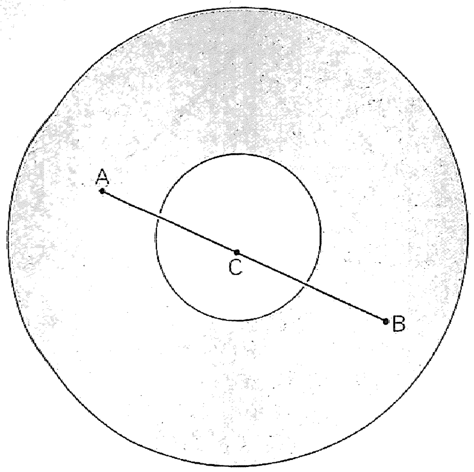


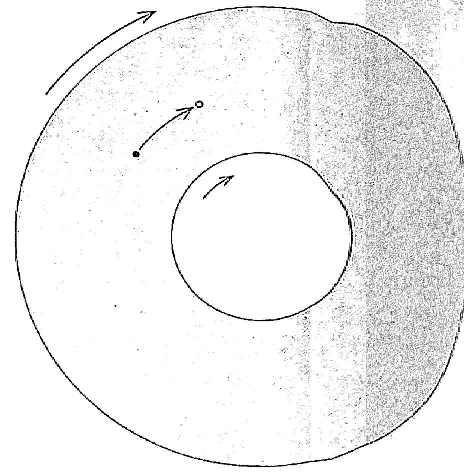
DESCRIPTION OF STRAIGHT STRING eight inches long is, by the same mathematical convention, the diagonal connecting the bottom left and top right corners of a square built on zero to eight on the horizontal axis (because the original and the "new" position of each point are the same). The intersection of the diagonal and the curve describing the folded string specifies a fixed point.



PROOF OF FIXED-POINT THEOREM depends on fact that every curve describing a folded string that is replaced uncut on top of a straight one must cross the diagonal describing the straight string. Two special cases are the curves of string with fixed point at right (*top*) and string with fixed point at left end (*bottom*). The theorem was set forth by the Dutch mathematician L. E. J. Brouwer.







CONVEXITY is one of two conditions a surface must satisfy if fixed-point theorems are to hold true on it (the other is boundedness). An area is convex if it contains every point on the line connecting any two of its points. The two circles at left do not form a convex surface; if one switches their position, the surface is transformed so that no fixed point remains. The ring (*second from left*)

rectangle such as our sheet of paper) are outlined in the illustration on the opposite page.

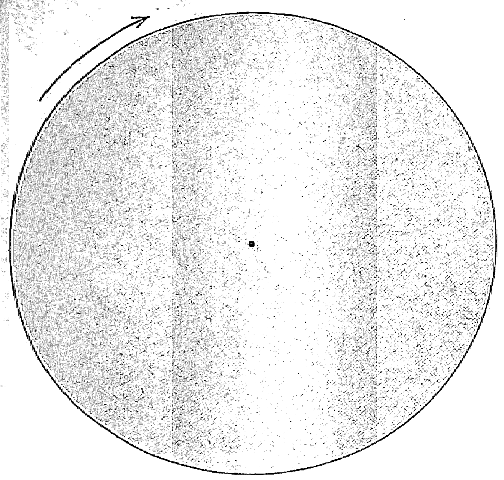
The Brouwer theorem does not apply to any area regardless of shape. An infinite domain, for example, need have no fixed point, even in one dimension. An infinitely long string can be moved in such a way that no point remains fixed. We need only to shift every point of the string one inch to the right. Since every point of the string has been moved an inch away from its original position, there is no fixed point. Hence we see that for an area always to have a fixed point when it is transformed, it must be bounded. It must also satisfy some other condition of shape, one that mathematicians call convexity. An area is defined as convex if it is possible to draw the line connecting any two points in the area so that no point of the line lies outside the area [*see illustration at top of these two pages*].

The Brouwer fixed-point theorem we have described as being applicable to one-dimensional and two-dimensional surfaces is in fact applicable to surfaces with any finite number of dimensions. The theorem does not hold, however, if the surface is infinite-dimensional. Fortunately there are fixed-point theorems that do apply in infinite-dimensional situations. We say "fortunately" because, surprising as it may seem, the greatest interest in fixed-point theorems is in the infinite-dimensional case. To understand why, let us consider Newton's famous second law of motion, which states that force is the product of mass and acceleration ($F = ma$). In most instances when the law is used, the force is a given function of the position of an object, and this position can always be found, given the acceleration of the object, by the techniques of calculus known as integration. Thus Newton's formula can be considered an equation for the position with the general form $f(x) = x$, where x denotes the position of the object and the known function, f, is determined by the forces, the masses and the initial positions and velocities. Fixed-point theorems are of great usefulness in helping us to understand equations of this type; indeed, a fixed-point theorem is usually cited in the proof that such equations have solutions.

Now consider the following question: Is it possible to put a satellite into a figure-eight orbit around both the earth and the moon? An affirmative answer amounts to saying that an equation $f(x) = x$ has a solution describing an orbit of the desired type. Any solution to such a problem is, of course, a function of time. It follows that we are trying to find if there is a function of time that satisfies the equation. The function $f(x)$ can be considered a transformation of functions of time into new functions of time in the same way that stirring coffee can be looked on as a transformation of points on a disk into new points. Accordingly the question becomes: Does the transformation represented by the function $f(x)$ have a fixed point? Such a function, since it is dependent on time, must be regarded as a "point" in an infinite-dimensional space. It is in trying to ascertain if such equations—equations involving unknown functions—have solutions of a given type that we require fixed-point theorems holding true even for infinite-dimensional surfaces.

Such questions of orbits can also be attacked by other methods; in fact, other methods, not involving fixed-point theorems directly, would normally be used to answer them. The most powerful methods of which we are aware, however, are those that appeal directly to fixed-point theorems in infinite-dimensional surfaces. It should come as no surprise, then, that there are many physical problems for which the only known method of solution involves fixed-point theorems. Problems of fluid flow are often of this type. Consider a stream bed with a bottom that rises and falls periodically like a sine curve [*see middle illustration on page 110*]. Is it possible for water to flow over this bottom in such a way that the surface of the water exhibits the same general periodicity as the bottom, or is every kind of flow necessarily nonperiodic? The answer is found to be that the surface can be periodic. This suggests a further question: Can the high points and low points of the surface occur directly above the high points and low points of the bottom, or must they be shifted slightly, either upstream or downstream? It has recently been demonstrated that there can be a flow with the high points of its surface lying directly over the high points of the bottom. There is no known way to show this without relying on high-powered fixed-point theorems, which cannot easily be visualized for cases involving simple surfaces such as a plane.

There is, however, one fixed-point theorem that can be readily described for finite-dimensional spaces and that remains valid in the infinite-dimensional case. Let us describe the theorem as it applies to a plane, which is of course a two-dimensional surface. Let P and Q represent points on the plane. If



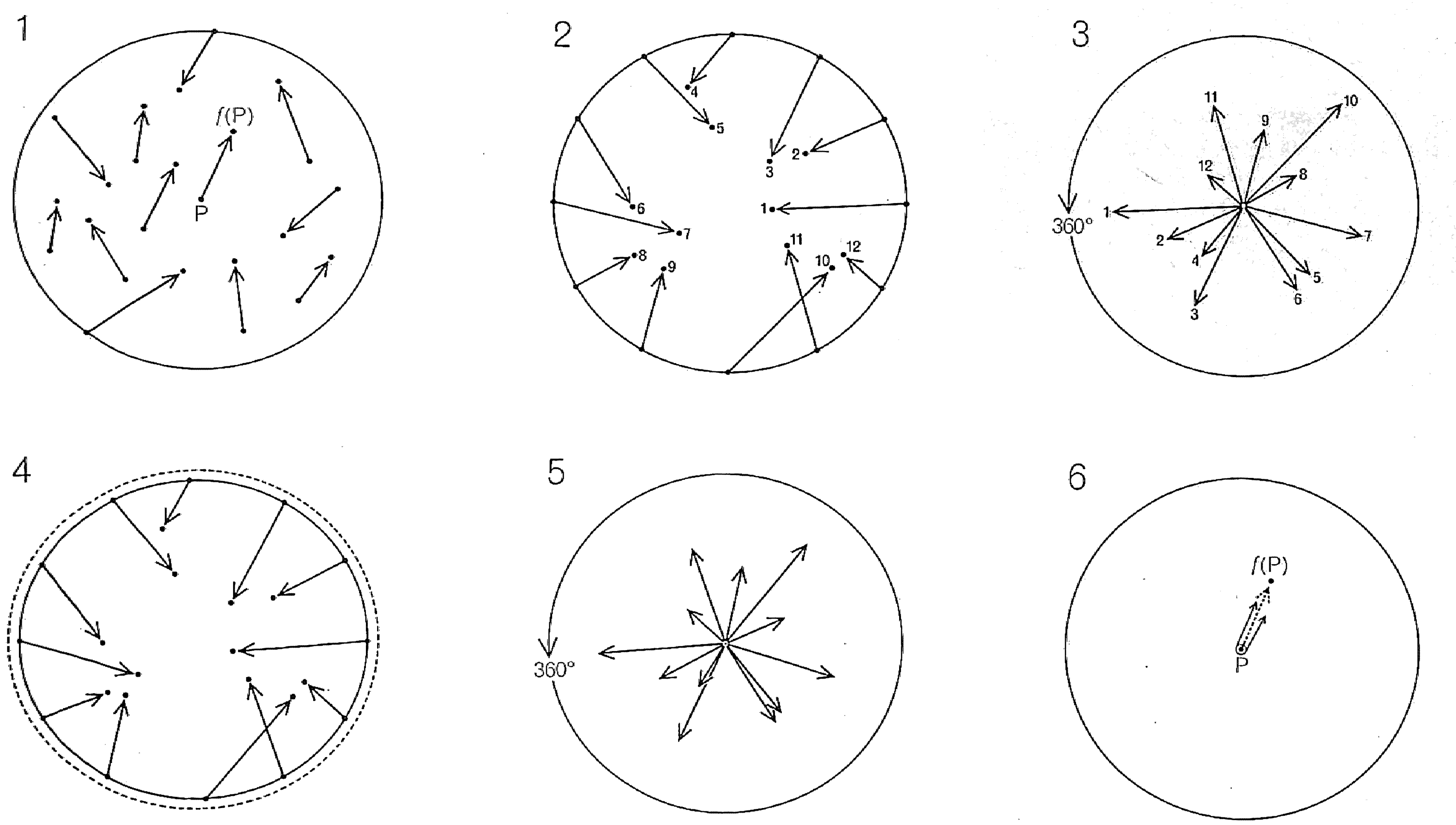
is not convex either, since rotation of the ring would cause every point on it to move. The circular disk at far right *is* convex.

the plane is transformed by stretching, twisting or folding part or all of it, the two points P and Q are transformed into new points that are determined by the deformation process and are therefore functions of P and Q. We denote this function by f, so that P is transformed into the point $f(P)$ and Q into $f(Q)$. If, following a certain transformation, the distance between the two points $f(P)$ and $f(Q)$ is always strictly smaller than the distance between the original points P and Q, then the transformation is called a contraction. There is a fixed-point theorem stating that every contraction has a fixed point; in other words, there must be a point in the same position before and after any contraction.

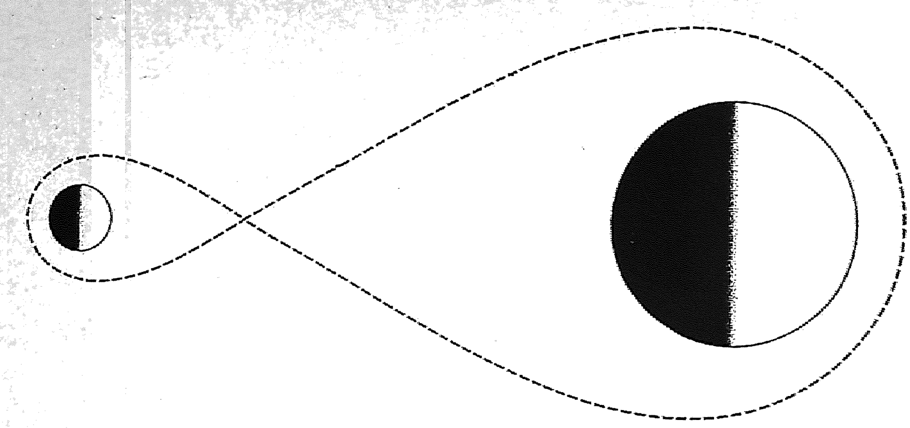
The proof of this theorem is not difficult to visualize [*see bottom illustration on next page*]. When a contraction takes place, any point P_1 on the original plane assumes a new position P_2. The point we have just designated P_2 occupies the spot originally occupied by a point that we say has moved to P_3. This point in turn now occupies the spot originally occupied by a point that we say has moved to P_4; and so on. Since we know that the transformation under consideration is a contraction, the distance between P_2 and P_3 must be smaller than the distance between P_2 and P_1. Similarly, the distance between P_4 and P_3 is smaller than the distance between P_3 and P_2, and so on. We obtain a sequence of points, $P_1, P_2, P_3 \ldots$, that get closer and closer together. This implies that the sequence must have a limit, which means only that all these points get closer and closer to some one point on the plane. This limiting point is a fixed point for the transformation.

The theorem for contractions has been stated and the idea of its proof has been outlined for transformations of a plane. In the preceding argument, however, the concept of dimension is never used. It follows that the theorem remains valid even in infinite-dimensional spaces whose "points" consist of functions of time.

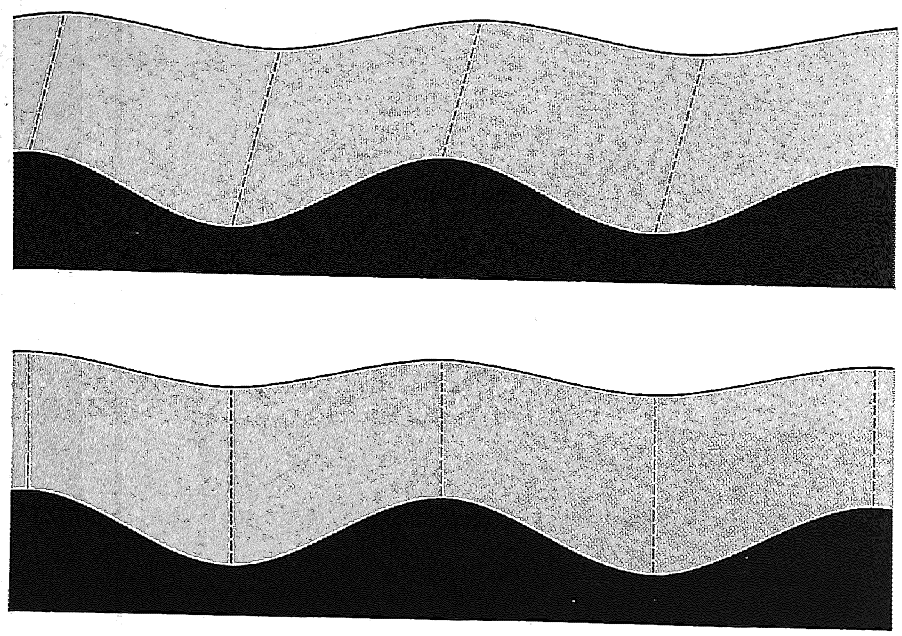
Not only does every contraction have a fixed point; it has only one fixed point. The proof of this is straightforward. Suppose P and Q were two different fixed points of the contraction $f(P)$. If this were the case, we should have $P = f(P)$ and $Q = f(Q)$. Now consider the distance between P and Q. Since these are fixed points, the dis-



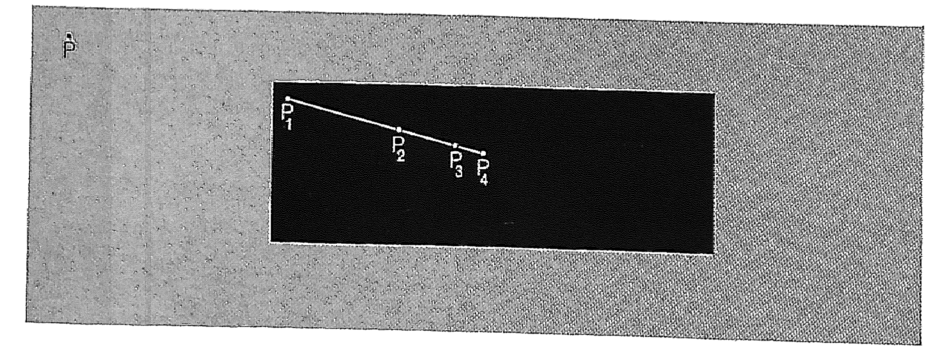
PROOF OF BROUWER'S THEOREM for two-dimensional surface such as a circular disk begins with the assumption, contrary to the theorem, that after deformation no point remains fixed. An arrow is drawn from each point to the position to which it is moved (*1*). Since no point is moved outside the disk all arrows from points on the boundary must head into the circle (*2*). These arrows are drawn again as if they emanated from a point within the circle (*3*). Considered thus, the arrows (called transformation vectors) make one complete rotation of 360 degrees around the circle. If we next trace the movement of points on the boundary of a concentric circle only slightly smaller than the original one (*4*), the number of rotations made by the arrows must, by the nature of continuous deformation, remain one (*5*). This must be true for all concentric circles because the rotation of the transformation vectors represents a continuous function. But when we consider a very small circle, all the arrows on its boundary head in roughly the same direction (*6*) and the net number of rotations is not one but zero. This contradiction shows that the assumption of no fixed point is untenable.



FEASIBILITY OF AN ORBIT by which a satellite would revolve around earth and moon is the type of question to which mathematicians apply fixed-point theorems for infinite-dimensional surfaces. The element of time in any equation for the orbit makes the problem infinite-dimensional, rendering such simple theorems as Brouwer's theorem inapplicable.



FEASIBILITY OF WATER FLOW of a certain type over a periodically rising and falling bottom can only be demonstrated by use of fixed-point theorems. Until recently it was not known if the surface of the water could rise and fall according to the same general period as the bottom. Now it has been demonstrated that such a flow is possible and that the high and low points of the surface can lie directly above the high and low points of the bottom.



CONTRACTION of a surface must result in one point remaining in the position it occupied before the contraction. The larger rectangle represents the original surface, a sheet of rubber stretched taut; the darker, smaller rectangle represents the sheet after it has sprung back to its relaxed position. We consider the point, P, near the corner at top left on the original rectangle. After the contraction it assumes a position we designate P_1. The point that was at P_1 originally has moved inward to a new position, P_2. The point originally at P_2 has moved to P_3, and so on. The interval between P_2 and P_3 is smaller than the interval between P_1 and P_2. In fact $P_1, P_2, P_3 \ldots$ form a series approaching a limit: the fixed point.

tance between them should be the same as the distance between $f(P)$ and $f(Q)$. But the distance between $f(P)$ and $f(Q)$ must, by the definition of a contraction, be strictly less than the distance between P and Q. This contradiction, calling for the distance between P and Q to be less than itself, shows that the original assumption that P and Q are two different fixed points is untenable and thus proves that there can be only one fixed point.

The fact that every contraction has a fixed point is customarily used to prove that differential equations (of which Newton's second law of motion, $F = ma$, is an example) have solutions. And, as we have seen, such equations can have only one solution. This suggests one highly practical consequence of the fixed-point theorems on contractions: in any mechanical system, whether it is the moon and the earth or a swinging pendulum, the motions of the system are completely determined by its initial displacements and velocities.

Much was made of this fact by the great French mathematician and astronomer Pierre Simon de Laplace. In his *Essai Philosophique sur les Probabilités* Laplace used it as the basis for commenting: "Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes." There has probably never been a more definite statement of the doctrine of predestination. It stood, seemingly irrefutable, for more than a century, until the theories of thermodynamics and quantum mechanics enabled it to be contradicted.

This discussion of fixed-point theorems serves to illustrate a phenomenon characteristic of mathematics. A purely geometric idea—the concept of a fixed point of a transformation of a plane or a line—has been generalized by analogy to apply to problems in mechanics and hydrodynamics and ultimately to the philosophical problem of predestination. Although it would be hard to maintain that all keys to philosophy lie in mathematics, it is true that modern mathematics, concerned with such interactions of geometric, algebraic and analytic ideas as we have described, does lend itself to philosophical applications.